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BAIRE CATEGORY PRINCIPLE AND UNIQUENESS THEOREM

BY

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ABSTRACT. Applying a theorem of Bagemihl and Seidel (1953), we prove that if S_2 is a set of second category in (α, β), where $0 \leqslant \alpha < \beta \leqslant 2\pi$, and if $f(z)$ is a function meromorphic in the sector $\Delta(\alpha, \beta) = \{z\colon 0 < |z| < \infty, \alpha < \arg z < \beta\}$ for which $\underline{\lim}_{r\to\infty}|f(re^{i\theta})| > 0$, for all $\theta \in S_2$, then there exists a sector $\Delta(\alpha', \beta') \subseteq \Delta(\alpha, \beta)$ such that $(\alpha', \beta') \subseteq \bar{S}_2$, S_2 is second category in (α', β'), and $f(z)$ has no zero in $\Delta(\alpha', \beta')$. Based on this property, we prove several uniqueness theorems.

1. Introduction. In [**1**, Theorem 2], Bagemihl and Seidel state and prove the following theorem: Let H_1 be a Hausdorff space, H_2 a Hausdorff space satisfying the second axiom of enumerability, and S a complete metric space. To every element $s \in S$, let there correspond a $\rho_s \subseteq H_1$ so that

(i) if $D \subseteq S$ is dense in an open set $G \subseteq S$, then ρ_D is dense in ρ_G.

Let f be a continuous function mapping H_1 into H_2 such that

(ii) if G is an open subset of S, then $f(\rho_G)$ is dense in H_2.

Then there exists a residual set $R \subseteq S$ such that for every $r \in R$, $f(\rho_r)$ is dense in H_2.

Now, let $H_1 = \Delta(\alpha, \beta) = \{z\colon 0 < |z| < \infty, \alpha < \arg z < \beta\}$, $H_2 = Z$, the extended z-plane, $S = [\alpha, \beta] = \{\theta\colon \alpha \leqslant \theta \leqslant \beta\}$, $\rho_\theta = \{z\colon 0 < |z| < \infty, \arg z = \theta\}$, $C_{\rho_\theta}(f, \infty) =$ the radial cluster set of f at infinity, $C_\Delta(f, \infty) =$ the angular cluster set of f at infinity, where $\Delta \subseteq \Delta(\alpha, \beta)$, and let f be meromorphic on $\Delta(\alpha, \beta)$. Then applying the above theorem, we obtain the following result (see [**1**, Theorem 9]).

THEOREM 1. *If $Z = C_\Delta(f, \infty)$ for every $\Delta \subseteq \Delta(\alpha, \beta)$, then there exists a residual set $R \subseteq [\alpha, \beta]$ such that $Z = C_{\rho_\theta}(f, \infty)$ for every $\theta \in R$.*

The above notations for cluster sets and the definitions of residual, first and second category can be found in Collingwood and Lohwater [**4**]. In the sequel, we are going to discuss the boundary properties at the point at infinity. Therefore, we restrict ourselves to the case that all functions $f(z)$ be holomorphic at the origin and $f(0) \neq 0$.

2. Category principles. We shall now apply Theorem 1 to study a relation between analyticity and category principles.

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THEOREM 2. *If S_2 is a set of second category in (α, β) and if $f(z)$ is a meromorphic function in $\Delta(\alpha, \beta)$ for which $\underline{\lim}_{r\to\infty}|f(re^{i\theta})| > 0$, for all $\theta \in S_2$, then there exists a sector $\Delta(\alpha', \beta') \subseteq \Delta(\alpha, \beta)$ such that $(\alpha', \beta') \subseteq \bar{S}_2$, S_2 is second category in (α', β'), and $f(z)$ has no zero in $\Delta(\alpha', \beta')$.*

PROOF. By the definition of second category, there exists a sector $\Delta(\alpha_0, \beta_0)$ such that $(\alpha_0, \beta_0) \subseteq \bar{S}_2$ and S_2 is everywhere of second category in (α_0, β_0) (see Kelley [9, p. 202]). Let $\{z_n\}$, $z_n = r_n e^{i\theta_n}$, be the sequence of all zeros of $f(z)$ in $\Delta(\alpha_0, \beta_0)$. If $\{\theta_n\}$ is not dense in (α_0, β_0), then clearly there is an interval $(\alpha', \beta') \subseteq (\alpha_0, \beta_0) \subseteq \bar{S}_2$ such that S_2 is second category in (α', β'), and $f(z)$ has no zero in $\Delta(\alpha', \beta')$. On the other hand, if $\{\theta_n\}$ is dense in (α_0, β_0), this leads to a contradiction.

By the restriction on $f(z)$, we know that this sequence $\{z_n\}$ tends to ∞. It follows that for every ray ρ_θ, $\theta \in S_2 \cap (\alpha_0, \beta_0)$, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that the distance $d(z_{n_k}, \rho_\theta) \to 0$, as $k \to \infty$. Inductively, we can choose $w_k \in \rho_\theta$ for which $d(z_{n_k}, w_k) = d(z_{n_k}, \rho_\theta)$. From the hypothesis, we find that $|f(w_k)| \geqslant c > 0$ for all sufficiently large k. Since $f(z_{n_k}) = 0$, $k = 1, 2, \ldots,$ it follows from a theorem of Bagemihl and Seidel [3, Theorem 2] that the function f cannot omit three values in any sector containing the ray ρ_θ, otherwise f would be normal in such a sector (see [3, p. 5]). Since the set S_2 is dense in (α_0, β_0), we conclude that $C_\Delta(f, \infty) = Z$ for every $\Delta \subseteq \Delta(\alpha_0, \beta_0)$. By applying Theorem 1, we find that there is a residual set $R \subset (\alpha_0, \beta_0)$ such that $C_{\rho_\theta}(f, \infty) = Z$ for every $\theta \in R$. This together with the hypothesis implies that $R \cap S_2 = \varnothing$, so that S_2 becomes a first category in (α_0, β_0), a contradiction. The proof is complete.

COROLLARY 1. *Under the hypothesis of Theorem 2, there exists a sector $\Delta(\alpha', \beta') \subseteq \Delta(\alpha, \beta)$ such that $(\alpha', \beta') \subseteq \bar{S}_2$, S_2 is second category in (α', β'), and $|f(z)| \geqslant c > 0$ in $\Delta(\alpha', \beta')$.*

PROOF. Suppose that the result is false. Then for any sector $\Delta \subseteq \Delta(\alpha, \beta)$, we would have the set $\{0\} \subseteq C_\Delta(f, \infty)$. By the same argument as in Theorem 2, we reach the same contradiction.

3. Uniqueness theorem. In this section, we shall prove some uniqueness theorems for functions bounded holomorphic in a sector. The present results are similar to earlier ones [8]. In [8, Theorems 1, 2], we gave a short proof of Shaginian's uniqueness theorem and a result of Erdös' conjecture for functions bounded holomorphic in the unit disk D. We now prove the following analogue of Shaginian's theorem which cannot be immediately derived from that of Shaginian.

THEOREM 3. *Let $g(z)$ be a bounded holomorphic function in $\Delta(-\alpha, \alpha)$ and let $A(r)$ be an increasing function satisfying $\lim_{r\to\infty} A(r) = \infty$. Suppose that*

$$|g(r)| \leqslant \exp(-A(r)r^{\pi/2\alpha}), \quad \text{for each } r > 0. \tag{1}$$

Then $g(z) \equiv 0$.

To prove this theorem, we need a lemma which is analogous to the lemma in [8].

LEMMA 1. *Let* $\omega(p, r) = \omega(p, [r, \infty); \Delta(-\alpha, \alpha) - [r, \infty))$ *be the harmonic measure at the point* p *of the continuum* $[r, \infty)$ *relative to the domain* $\Delta(-\alpha, \alpha) - [r, \infty)$, *where* $0 < p < r$. *Then*

$$\lim_{r\to\infty} r^{\pi/2\alpha}\omega(p, r) = (2/\pi)p^{\pi/2\alpha}.$$

PROOF. The method we use is based on a theorem of Nevanlinna [**10**, p. 106]. Let $w(z) = z^{\pi/2\alpha}$ be the conformal mapping from $\Delta(-\alpha, \alpha)$ onto $H = \{w: \operatorname{Re} w > 0\}$. Then by the conformal invariance of harmonic measure [**10**, p. 38] we have

$$\omega(p, r) = \omega\left(p^{\pi/2\alpha}, r^{\pi/2\alpha}\right). \tag{2}$$

Let $\zeta(w)$ be the conformal mapping from H onto D defined by

$$\zeta(w) = (w - r^{\pi/2\alpha})/(w + r^{\pi/2\alpha}).$$

Then we have

$$\omega\left(p^{\pi/2\alpha}, r^{\pi/2\alpha}\right) = \omega\left(\zeta(p^{\pi/2\alpha}), [0, 1)\right). \tag{3}$$

According to Nevanlinna's theorem, (2) and (3) give

$$\omega(p, r) = \frac{2}{\pi}\sin^{-1}\frac{1 + \zeta(p^{\pi/2\alpha})}{1 - \zeta(p^{\pi/2\alpha})} = \frac{2}{\pi}\sin^{-1}\left(\frac{p}{r}\right)^{\pi/2\alpha}.$$

This implies that

$$\lim_{r\to\infty} r^{\pi/2\alpha}\omega(p, r) = \frac{2}{\pi}\lim_{r\to\infty} r^{\pi/2\alpha}\sin^{-1}\left(\frac{p}{r}\right)^{\pi/2\alpha} = \frac{2}{\pi}p^{\pi/2\alpha}.$$

PROOF OF THEOREM 3. Without loss of generality, we may assume that $|g(z)| \leqslant 1$ in $\Delta(-\alpha, \alpha)$. Then by hypothesis (1) and the two constants theorem [**10**, p. 42], we have

$$|g(p)| \leqslant \exp(-A(r)r^{\pi/2\alpha}\omega(p, r)), \quad \text{where } 0 < p < r. \tag{4}$$

Let ε be given with $0 < \varepsilon < (2/\pi)p^{\pi/2\alpha}$. Then by Lemma 1 there is an r_0 such that

$$r^{\pi/2\alpha}\omega(p, r) > (2/\pi)p^{\pi/2\alpha} - \varepsilon, \quad \text{for } r \geqslant r_0. \tag{5}$$

Combining (3) and (4), we obtain

$$|g(p)| \leqslant \exp\left(-A(r)(2p^{\pi/2\alpha}/\pi - \varepsilon)\right) \quad \text{for } r \geqslant r_0.$$

Now, apply the condition $\lim_{r\to\infty} A(r) = \infty$, and we find from the above inequality that $g(p) = 0$. Since $p > 0$ is arbitrary, it follows that $g(z) \equiv 0$ on account of the classical uniqueness theorem.

COROLLARY 2. *Let* $g(z)$ *be a bounded holomorphic function in* $\Delta(-\alpha, \alpha)$ *satisfying* $|g(r)| \leqslant \exp(-r^{\pi/2\alpha+\varepsilon})$, $\varepsilon > 0$, *for all sufficiently large* r; *then* $g(z) \equiv 0$.

PROOF. Let $A(r) = r^{\varepsilon}$; then $\lim_{r\to\infty} A(r) = \infty$. The result now follows immediately from Theorem 3.

4. Category principles and uniqueness theorem. With the help of Theorem 3 we are now able to prove the following result based on category principles.

THEOREM 4. *If S_2 is a set of second category in (α, β) and if $f(z)$ is a meromorphic function in $\Delta(\alpha, \beta)$ for which*

$$\varliminf_{r\to\infty} |f(re^{i\theta})|/e^{r^n} > 0, \quad \textit{for all } \theta \in S_2, \tag{6}$$

then there exists a positive number N which depends on the funcion f and the set S_2 such that $n > N$ implies $f(z) \equiv \infty$.

PROOF. According to (6), we clearly have

$$\varliminf_{r\to\infty} |f(re^{i\theta})| > 0, \quad \text{for all } \theta \in S_2.$$

It follows from Corollary 1 that there is a sector $\Delta(\alpha', \beta') \subseteq \Delta(\alpha, \beta)$ and a positive number c such that

$$(\alpha', \beta') \subseteq \bar{S}_2 \quad \text{and} \quad |f(z)| \geqslant c, \quad \text{for all } z \in \Delta(\alpha', \beta'). \tag{7}$$

By a rotation, we may, without loss of generality, assume that $(\alpha', \beta') = (-\alpha_0, \alpha_0) \subseteq \bar{S}_2$ and the point $0 \in S_2$.

Again, from (6), there are two positive numbers c' and r_0 such that

$$1/|f(r)| \leqslant c' \exp(-r^n), \quad \text{for } r \geqslant r_0. \tag{8}$$

Let $g(z) = 1/(c'f(z))$. Then by (7) and (8) we find that

$$|g(z)| \leqslant 1/(cc'), \quad \text{for } z \in \Delta(-\alpha_0, \alpha_0) \quad \text{and}$$
$$|g(r)| \leqslant \exp(-r^n), \quad \text{for } r \geqslant r_0.$$

By choosing $N = \pi/(2\alpha_0)$ and applying Corollary 2, we conclude that $g(z) \equiv 0$ or $f(z) \equiv \infty$ provided $n > N$. This completes the proof.

We remark that at the beginning we restricted ourselves to the case that all functions $f(z)$ be holomorphic at $z = 0$ and $f(0) \neq 0$. Symmetrically, if we require that all functions $f(z)$ be holomorphic at $z = \infty$ and $f(\infty) \neq 0$, then we have the following symmetric results.

THEOREM 5. *Under the hypothesis of Theorem* 4, *if* (6) *is replaced by*

$$\varliminf_{r\to 0} |f(re^{i\theta})|/e^{r^{-n}} > 0, \quad \textit{for all } \theta \in S_2,$$

then there exists $N > 0$ such that $n > N$ implies $f(z) \equiv \infty$.

THEOREM 6. *Under the hypothesis of Theorem* 4, *if* (6) *is replaced by*

$$\varlimsup_{r\to\infty} |f(re^{i\theta})|/e^{-r^n} < \infty, \quad \textit{for all } \theta \in S_2,$$

then there exists $N > 0$ such that $n > N$ implies $f(z) \equiv 0$.

THEOREM 7. *Under the hypothesis of Theorem* 4, *if* (6) *is replaced by*

$$\varlimsup_{r\to 0} |f(re^{i\theta})|/e^{-r^{-n}} < \infty, \quad \textit{for all } \theta \in S_2,$$

then there exists $N > 0$ such that $n > N$ implies $f(z) \equiv 0$.

The general meaning of the above four theorems claims that if $|f(re^{i\theta})|$ tends to a limit a, $0 \leqslant a \leqslant \infty$, on a set of second category, then the speed is always less than e^{r^n} for some $n > 0$. This number n depends both on the function $f(z)$ and the set S_2. Theorem 2 allows us to estimate an upper bound for n, i.e. $n \leqslant \pi/(\beta' - \alpha')$ where $(\alpha', \beta') \subseteq \bar{S}_2$ and $\Delta(\alpha', \beta')$ is one of the sectors in $\Delta(\alpha, \beta)$ such that $f(z)$ is bounded below in $\Delta(\alpha', \beta')$. The number $\beta' - \alpha'$ is the opening angle of the sector $\Delta(\alpha', \beta')$. If we denote by δ the biggest opening angle having the above property, then $n \leqslant \pi/\delta$.

5. Uniqueness theorem for harmonic functions. We notice that the holomorphic function $f(z)$ in Theorem 3 can be replaced by a subharmonic function $h(z)$ because the maximum principle is true for subharmonic functions. Moreover, if $h(z)$ is harmonic then the function $f(z) = \exp(h(z) + i\bar{h}(z))$ is holomorphic, where $\bar{h}(z)$ is a harmonic conjugate of $h(z)$. Therefore the speed e^{r^n} can be replaced by the speed r^n.

THEOREM 8. *Let $h(z)$ be a harmonic function bounded above in $\Delta(-\alpha, \alpha)$ and let $A(r) \to \infty$, as $r \to \infty$. If*

$$h(r) \leqslant -A(r)r^{\pi/2\alpha}, \quad \text{for each } r > 0,$$

then $h(z) \equiv -\infty$.

PROOF. By what we just remarked that the function $f(z) = \exp(h(z) + i\bar{h}(z))$ be bounded holomorphic in $\Delta(-\alpha, \alpha)$, we find that $|f(r)| = e^{h(r)} \leqslant \exp(-A(r)r^{\pi/2\alpha})$ for each $r > 0$. It follows from Theorem 3 that $f(z) \equiv 0$ or $h(z) \equiv -\infty$.

By using the same argument as in the proof of Theorem 8 and the assertion of Theorem 4, we can easily obtain the following uniqueness theorem for harmonic functions.

THEOREM 9. *If S_2 is a set of second category in $(-\alpha, \alpha)$ and if $h(z)$ is a harmonic function in $\Delta(-\alpha, \alpha)$ for which*

$$\varliminf_{r\to\infty} |h(re^{i\theta})|/r^n > 0, \quad \text{for all } \theta \in S_2,$$

then there exists $N > 0$ such that $n > N$ implies $h(z) \equiv \infty$.

To conclude this section, let us state the following uniqueness theorem of Erdös' conjecture [**8**, Theorem 2] and its corresponding result for harmonic functions in the unit disk as follows.

THEOREM 10. *Let $f(z)$ be bounded holomorphic in $D = \{z: |z| < 1\}$ such that*

$$\min_{|z|=r} |f(z)| \leqslant \exp(-A(1-r)/(1-r)), \quad \text{for each } 0 < r < 1,$$

where $\lim_{r\to 1} A(1-r) = \infty$; then $f(z) \equiv 0$.

By the same argument as in Theorem 8, we obtain immediately

THEOREM 11. *Let $h(z)$ be a harmonic function bounded above in D such that*

$$\min_{|z|=r} h(z) \leqslant -A(1-r)/(1-r), \quad \text{for each } 0 < r < 1,$$

where $\lim_{r\to 1} A(1-r) = \infty$; then $h(z) \equiv -\infty$.

Notice that the above uniqueness theorem of Erdös' conjecture was proved earlier by Heins [**6**, Theorem 7.1] which was not known to either Erdös or us in 1970.

6. Uniqueness theorem in a half-plane. In this section, we shall extend Theorem 10 from the disk D to a half-plane, say, $H = \{z\colon \operatorname{Re} z > 0\}$. Extension of this kind cannot be directly achieved by a conformal mapping from D onto H. The main difficulty occurs in "the contraction principle" due to Heins [**6**, p. 182]. More precisely, let $\{z_n\}$ be a sequence of points in D and let B^*, b^* be two associated Blaschke products defined by

$$\begin{cases} B^*(z) = \prod_{n=1}^{\infty} \dfrac{z_n - z}{1 - \bar{z}_n z} \cdot \dfrac{\bar{z}_n}{|z_n|}, \quad \text{and} \\ b^*(z) = \prod_{n=1}^{\infty} \dfrac{|z_n| + z}{1 + |z_n| z}, \quad \text{where } \sum_{n=1}^{\infty} (1 - |z_n|) < \infty. \end{cases} \tag{9}$$

For convenience, we shall say that the product b^* is the contraction function of B^*. Denote the maximum and minimum modulus of a function f, respectively, by

$$M(r;f) = \operatorname*{Max}_{|z|=r} |f(z)| \quad \text{and} \quad m(r;f) = \min_{|z|=r} |f(z)|.$$

Then the contraction principle asserts that

$$M(r;B^*) \leqslant M(r;b^*) = b^*(r)$$

and

$$|b^*(-r)| = m(r;b^*) \leqslant m(r;B^*).$$

In other words, the contraction function b^* increases the maximum but decreases the minimum modulus of B^*. This principle is in fact derived from Harnack's inequality.

We now let $\{a_n\}$ be a sequence of points in H and let it be divided into two subsequences $\{a_j\}$ and $\{a_k\}$ such that $\{a_j\}$ is bounded and $\{a_k\} \to \infty$, as $k \to \infty$. If the series

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re} a_n}{1 + |a_n|^2} < \infty, \tag{10}$$

then the associated Blaschke product B (see Hille [7, p. 457]) is defined by

$$B(z) = \prod_{j=1}^{\infty} \frac{a_j - z}{\bar{a}_j + z} \cdot \prod_{k=1}^{\infty} \frac{a_k - z}{\bar{a}_k + z} \cdot \frac{\overline{a_k}}{a_k}. \tag{11}$$

In contrast to the contraction function defined in (9), we may also define the following contraction function of B:

$$b(z) = \prod_{j=1}^{\infty} \frac{|a_j| - z}{|a_j| + z} \cdot \prod_{k=1}^{\infty} \frac{|a_k| - z}{|a_k| + z} = \prod_{n=1}^{\infty} \frac{|a_n| - z}{|a_n| + z}, \tag{12}$$

where the associated series

$$\sum_{n=1}^{\infty} \frac{|a_n|}{1 + |a_n|^2} < \infty. \tag{13}$$

Notice that the main difficulty concerning the contraction principle is that the convergence of the series $\Sigma(1 - |z_n|)$ guarantees both products B^* and b^* to be well defined in (9). However, the same property cannot hold for B and b. If we have no restriction on the sequence $\{a_n\}$, then the convergence of the series in (10) cannot guarantee that of (13). For instance, if $a_n = 1 + ni$, then clearly the series in (10) converges while the series in (13) diverges. In this case, the sequence $\{a_n\}$ tends to infinity tangentially. This gives the motivation to formulate the following uniqueness theorem.

THEOREM 12. *Let $f(z)$ be a bounded holomorphic function in H and let $A(r)$ be an increasing function satisfying* $\lim_{r\to\infty} A(r) = \infty$, *and*

$$\min_{|\theta|\leq\pi/2-\delta} |f(re^{i\theta})| \leq \exp(-A(r)r), \quad \textit{for each } r > 0 \textit{ and some } 0 < \delta < \pi/2. \tag{14}$$

If $\{a_n\} = \{a_j\} \cup \{a_k\}$ is the sequence of zeros of f such that $\{a_j\}$ is bounded and $\{a_k\}$ tends to infinity, then the function $f(z) \equiv 0$, provided either the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty, \tag{15}$$

or the sequence $\{a_k\}$ tends to infinity nontangentially.

As before, we denote the minimum and maximum modulus of a function f by

$$m(r;f) = \inf_{|\theta|<\pi/2} |f(re^{i\theta})|; \quad M(r;f) = \sup_{|\theta|<\pi/2} |f(re^{i\theta})|.$$

To prove Theorem 12, we shall first state and prove the following different type of contraction principle which describes that the minimum of the contraction function occurs on the positive real axis.

LEMMA 2. *Let $\{a_n\}$ be a sequence of points in H satisfying condition* (13), *and let B, b be two associated Blaschke products defined by* (11) *and* (12). *Then we have*

$$|b(r)| = m(r;b) \leq m(r;B) \quad \textit{and} \quad M(r;B) \leq M(r;b) = |b(-r)|. \tag{16}$$

PROOF. From the triangle inequality, we obtain the following different type of Harnack's inequality:

$$\left|\frac{|a_n| - |z|}{|a_n| + |z|}\right| \leq \left|\frac{a_n - z}{\bar{a}_n + z}\right|, \quad \text{for each } n.$$

By multiplying all terms together, we conclude the first inequality of (16). The same argument yields the second inequality of (16). This proves the lemma.

PROOF OF THEOREM 12. According to the representation of bounded holomorphic functions (see Hille [7, p. 457]), the function $f(z)$ can be represented by

$$f(z) = B(z)E(z), \quad E(z) = \exp\left(-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dq(t)}{z - it} + c\right), \tag{17}$$

where B is the Blaschke product defined by (11) which contains all zeros $\{a_n\}$ of f, the function q is a monotone increasing function of bounded variation in $(-\infty, \infty)$, and the number c is a constant.

We shall first prove that the inverse function $E^{-1}(z)$ of $E(z)$ satisfies the following condition for some constant $d > 0$:

$$|E^{-1}(re^{i\theta})| \leqslant e^{dr}, \quad \text{for each } r \geqslant 2 \text{ and } |\theta| \leqslant \pi/2 - \delta. \tag{18}$$

From the property of the function q, we can see that

$$0 < \int_{-\infty}^{\infty} dq(t) = q(\infty) - q(-\infty) < \infty. \tag{19}$$

In view of (17), we find that

$$|E^{-1}(re^{i\theta})| = e^{I(r,\theta)},$$

where

$$\begin{aligned} I(r,\theta) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{r\cos\theta\, dq(t)}{r^2 - 2rt\sin\theta + t^2} \\ &< \frac{r}{\pi}\int_{-\infty}^{\infty} \frac{dq(t)}{(r-t)^2 + 2rt(1-\cos\delta)} \\ &< \frac{r}{\pi}\int_{-\infty}^{r-1} \frac{dq(t)}{(r-t)^2} + \frac{r}{\pi}\int_{r-1}^{r+1} \frac{dq(t)}{2rt(1-\cos\delta)} + \frac{r}{\pi}\int_{r+1}^{\infty} \frac{dq(t)}{(r-t)^2} \\ &< \frac{r}{\pi}\left(1 + \frac{1}{2r(r-1)(1-\cos\delta)}\right)\int_{-\infty}^{\infty} dq(t). \end{aligned} \tag{20}$$

Since $r \geqslant 2$, by choosing the constant

$$d = \frac{1}{\pi}\left(1 + \frac{1}{4(1-\cos\delta)}\right)(q(\infty) - q(-\infty)),$$

we conclude the assertion (18) from (19) and (20).

To prove the assertion, we consider the first case that the series in (15) is convergent. This implies that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{1 + |a_n|^2} < \sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty,$$

and therefore the contraction function b of B described in (12) is well defined. By Lemma 2, we find that

$$|b(r)| = m(r; b) \leqslant m(r; B). \tag{21}$$

Now, from (14), (17), and (18), we have

$$\begin{aligned} m(r;B) = m(r; fE^{-1}) &\leqslant \min_{|\theta| \leqslant \pi/2 - \delta} |f(re^{i\theta})E^{-1}(re^{i\theta})| \\ &\leqslant \exp(-(A(r) - d)r), \quad \text{for each } r \geqslant 2. \end{aligned} \tag{22}$$

Combining (21) and (22), we obtain

$$|b(r)| \leqslant \exp(-(A(r) - d)r), \quad \text{for each } r \geqslant 2.$$

By applying Theorem 3 for $\alpha = \pi/2$, we conclude that $b(z) \equiv 0$, so that $B(z) = f(z) \equiv 0$.

Finally, we consider the second case that the sequence $\{a_k\}$ tends to infinity nontangentially. In this case, the series in (13) can possibly be divergent so that the contraction function b of B is not well defined. However, the same argument can be used by considering the subproduct of B. In view of (11), we denote B_1 and B_2 to be the first and second product there. Since the first sequence $\{a_j\}$ is bounded, hence the first product B_1 satisfies

$$\lim_{z \to \infty} |B_1(z)| = 1, \quad \text{where } z \in H. \tag{23}$$

We shall now prove that the contraction function b_2 of B_2 is well defined. To do this, we write $a_k = b_k + ic_k$. Since this sequence $\{a_k\}$ tends to infinity nontangentially, there is a positive number m such that

$$|c_k| \leqslant mb_k, \quad \text{for each } k = 1, 2, \ldots. \tag{24}$$

Since $\{a_k\}$ is a subsequence of $\{a_n\}$, it follows from (10) and (24) that

$$\sum_{k=1}^{\infty} \frac{|a_k|}{1 + |a_k|^2} \leqslant (1 + m^2)^{1/2} \sum_{k=1}^{\infty} \frac{\operatorname{Re} a_k}{1 + |a_k|^2} < \infty.$$

This concludes that the contraction function b_2 of B_2 is well defined.

In view of (17), we have

$$B_2(z) = f(z)E^{-1}(z)B_1^{-1}(z), \quad \text{for } z \in H.$$

By using (22), (23), and Lemma 2, we finally obtain

$$|b_2(r)| \leqslant (1 - \varepsilon)\exp(-(A(r) - d)r),$$

for all sufficiently large r and some $0 < \varepsilon < 1$. This yields the assertion.

7. Some examples. Finally, we want to give some examples to explain the necessary conditions in our results. Let us start from Theorem 2. We want to show that the condition of second category of the set S_2 in (α, β) is necessary. Namely, Theorem 2 is false if we replace S_2 by S_1 which is an F_σ set of first category relative to (α, β).

EXAMPLE 1. There exists an entire function $f(z)$ for which

$$\underline{\lim}_{r \to \infty} |f(re^{i\theta})| > 0, \quad \text{for all } \theta \in S_1 \subseteq (\alpha, \beta) \tag{25}$$

and $f(z)$ has always zeros in any sector $\Delta \subseteq \Delta(\alpha, \beta)$.

PROOF. Clearly we may assume that $\bar{S}_1 = [\alpha, \beta]$. The set $[\alpha, \beta] - S_1$ is a residual set. We choose a sequence $S_1^* = \{\theta_n\}$, $\theta_n \in (\alpha, \beta) - S_1$ such that $\overline{S_1^*} = [\alpha, \beta]$. Clearly the union $E = S_1 \cup S_1^*$ is again an F_σ set of first category relative to (α, β). Let

$$z_n = ne^{i\theta_n}, \quad D_n = \{z : |z - z_n| < 1/n\},$$

and define the function

$$\begin{aligned} g(z) &= c, \quad \text{for all } z \in \Delta(\alpha, \beta) - \bigcup_{n=1}^{\infty} D_n \\ &= 0, \quad \text{for } z = z_n, n = 1, 2, \ldots. \end{aligned}$$

Then by the Tietze-Urysohn theorem (see Kelley [9]), $g(z)$ can be extended to be continuous in $\Delta(\alpha, \beta)$. According to [2, Theorem 1], there exists an entire function $f(z)$ such that the cluster set of f coincides with that of $g(z)$, so that (25) holds.

Moreover, from Walsh's lemma [**11**, p. 310] it is easy to see that the function $f(z)$ can be chosen to have the same value as $g(z)$ at the sequence, i.e. $f(z_n) = g(z_n) = 0$. Hence the function $f(z)$ has zeros in any sector $\Delta \subseteq \Delta(\alpha, \beta)$.

The next example will show that the condition of the function $f(z)$ is also necessary. As we pointed out before the function $f(z)$ in Theorem 2 can be replaced by a subharmonic function. We shall show that it cannot be replaced by a continuous function.

EXAMPLE 2. There exists a continuous function $f(z)$ for which

$$\lim_{r\to\infty} |f(re^{i\theta})| = c, \quad \text{for all } \theta \in (\alpha, \beta)$$

and $f(z)$ has always zeros in any sector $\Delta \subseteq \Delta(\alpha, \beta)$.

PROOF. Let $A = \{e^{i\theta}: \alpha < \theta < \beta\}$, $B = \{re^{i\theta}: \frac{1}{2} < r < 2, \alpha < \theta < \beta\}$, and let

$$\begin{aligned} g(z) &= 0 \quad \text{on } A \\ &= c \quad \text{on } \Delta(\alpha, \beta) - B. \end{aligned}$$

Again by the Tietze-Urysohn theorem $g(z)$ can be extended to $f(z)$ which is continuous and satisfies the required property.

The minimum growth conditions in the previous uniqueness theorems are necessary as will be seen from the following:

EXAMPLE 3. The holomorphic function $f(z) = \exp(-z^{\pi/2\alpha})$ and the harmonic function $h(re^{i\theta}) = -r^{\pi/2\alpha}\cos(\pi\theta/2\alpha)$ are sufficient to verify the necessity in Theorems 3, 8, and 12. Also the superharmonic function $h(re^{i\theta}) = -r^{\pi/2\alpha+\varepsilon}$, $\varepsilon > 0$, shows that the condition of harmonicity in Theorem 8 cannot be replaced by superharmonicity.

Finally, we shall show that the condition of second category cannot be replaced by first category in Theorems 4–7. This needs to be verified only for Theorem 4. The construction here is based upon the tangential approximation of Arakélian (see Gauthier and Seidel [**5**, Theorems 1.11, 2.3]).

EXAMPLE 4. If S_1 is a set of first category in $(0, 2\pi)$, then there is an entire function $f(z)$ such that for any $n > 0$,

$$\lim_{r\to\infty} |f(re^{i\theta})|/e^{r^n} = \infty, \quad \text{for all } \theta \in S_1.$$

PROOF. From the hypothesis of S_1, we may write $S_1 = \bigcup_{k=1}^{\infty} T_k$, where T_k is closed and nowhere dense in $[0, 2\pi]$. We set $S_1^* = \bigcup_{k=1}^{\infty} T_k^*$, where $T_k^* = \{re^{i\theta}: k \leq r < \infty \text{ and } \theta \in T_k\}$. Then S_1^* satisfies the hypothesis of Arakélian's theorem.

Again, by the Tietze-Urysohn theorem, there is a continuous function $g(z)$ such that

$$g(re^{i\theta}) = e^{e^r}, \quad \text{for } re^{i\theta} \in S_1^*.$$

Let $\varepsilon(r)$ be a positive function satisfying $\varepsilon(r) \to 0$, as $r \to \infty$. Then by Arakélian's theorem, there is an entire function $f(z)$ for which

$$|f(z) - g(z)| < \varepsilon(|z|), \quad \text{for } z \in S_1^*;$$

equivalently,

$$|f(re^{i\theta})| > e^{e^r} - \varepsilon(r), \quad \text{for } re^{i\theta} \in S_1^*.$$

This yields the following assertion:

$$\lim_{r\to\infty} |f(re^{i\theta})| \geqslant \lim_{r\to\infty} e^{e^r - r^n} = \infty, \quad \text{for all } \theta \in S_1.$$

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